TWO EXACT SOLUTIONS OF THE NAVIER–STOKES EQUATIONS FOR THE CONCENTRATION OF EDDIES IN A VISCOUS LIQUID

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In the final analysis, vorticity in a liquid or gas is broken down by viscosity [1]; however, there are known cases of the appearance and long-term existence of three-dimensional eddies in water, air, and other media. Therefore, the conditions under which vorticity can even rise with viscosity are of interest. For example, with the flow of a liquid out of an opening in the bottom of a rotating cylindrical vessel, the total momentum with respect to the vertical axis of the vessel increases with the time [2, 3]. For some flows, there exist contradictory opinions: In [4, 5] it is asserted that an eddy around a flat sink in a viscous liquid is damped, while, in [6, 7], it is argued that, with determined Reynolds numbers, there is an increase in the vorticity around a sink. The present article gives exact solutions of the Navier–Stokes equations, demonstrating the development of eddies in a viscous liquid.

§1. There is great interest in the case where, from the flow of a viscous liquid with a vortex $|\boldsymbol{\Omega}|$ close to zero, a flow with a finite vorticity is formed. Therefore, we connect the statement of the problem to the question of the stability of the flow of a viscous liquid originally having $\boldsymbol{\Omega} = 0$, with respect to determined perturbations. Using the example of a problem with axial symmetry in a cylindrical system of coordinates (r, θ, z), we shall demonstrate a method for combining a search for exact solutions of the Navier–Stokes equations with a consideration of the stability of the flow of a viscous liquid. The main flow, for which the components of the vortex are equal to zero, has the form

$$\begin{cases} \Omega_\theta = \gamma = 0, \\ v_r = \dfrac{\partial\Phi}{\partial r}, \\ v_z = \dfrac{\partial\Phi}{\partial z}, \end{cases} \tag{1.1}$$

where $\gamma = 2\pi r v_\theta$ is the circulation of the velocity; Φ is the potential for the components of the velocities v_r and v_z. One main flow or another can be obtained by the choice of the potential Φ.

The Navier–Stokes equations and the continuities with axial symmetry are represented in the form

$$\begin{cases} \dfrac{\partial\Omega_\theta}{\partial t} + v_r\left(\dfrac{\partial\Omega_\theta}{\partial r} - \dfrac{\Omega_\theta}{r}\right) + v_z\dfrac{\partial\Omega_\theta}{\partial z} = \dfrac{1}{r}\dfrac{\partial}{\partial z}\left(\dfrac{\gamma}{2\pi r}\right)^2 + \\ \qquad + \nu\left[\dfrac{\partial}{\partial r}\left(\dfrac{1}{r}\dfrac{\partial(r\Omega_\theta)}{\partial r}\right) + \dfrac{\partial^2\Omega_\theta}{\partial z^2}\right], \\ \dfrac{\partial\gamma}{\partial t} + v_r\dfrac{\partial\gamma}{\partial r} + v_z\dfrac{\partial\gamma}{\partial z} = \nu\left[r\dfrac{\partial}{\partial r}\left(\dfrac{1}{r}\dfrac{\partial\gamma}{\partial r}\right) + \dfrac{\partial^2\gamma}{\partial z^2}\right], \\ \dfrac{\partial v_r}{\partial z} - \dfrac{\partial v_z}{\partial r} = \Omega_\theta, \quad \dfrac{\partial(rv_r)}{\partial r} + \dfrac{\partial(rv_z)}{\partial z} = 0. \end{cases} \tag{1.2}$$

To remain within the framework of the Navier–Stokes equations, out of all the possible perturbations to the main flow (1.1) only those perpendicular to the flow must be taken, i.e., the perturbation γ. In addition, we require that the value of γ not depend on z; we separate the first two equations of the system (1.2). The final equation for the perturbation γ has the form

$$\frac{\partial\gamma}{\partial t} + \frac{\partial\Phi}{\partial r}\frac{\partial\gamma}{\partial r} = \nu r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\gamma}{\partial r}\right). \tag{1.3}$$

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§2. The problem of the rotation of the liquid around a flat source is obtained from the general problem if $\Phi = (m/2\pi)\ln r$ is substituted into formula (1.3), where m is the abundance of the source. Then Eq. (1.3) is transformed to the form

$$\frac{1}{\nu}\frac{\partial \gamma}{\partial t} + (\mathrm{Re}+1)\,\frac{1}{r}\frac{\partial \gamma}{\partial r} = \frac{\partial^2 \gamma}{\partial r^2}, \tag{2.1}$$

where $\mathrm{Re} = m/(2\pi\nu)$ is the Reynolds number.

Steady-state solutions of Eq. (2.1) have been examined in detail in [6]. There exist several methods for obtaining non-steady-state solutions. A characteristic non-steady-state solution which, more than any other, brings out contradictions, is the solution of the problem of rectilinear vortex diffusion in the presence of a sink. All the methods, used in the majority of papers give the following solution of the problem:

$$\begin{cases} v_\theta = \dfrac{\gamma_\infty}{2\pi r}\left\{1 - \dfrac{\Gamma\left(\dfrac{\mathrm{Re}+2}{2};\ \dfrac{r^2}{4\nu t}\right)}{\Gamma\left(\dfrac{\mathrm{Re}+2}{2}\right)}\right\}, \\ \Omega_z = \dfrac{1}{r}\dfrac{\partial (r v_\theta)}{\partial r} = \dfrac{\gamma_\infty}{\pi r^2 \Gamma\left(\dfrac{\mathrm{Re}+2}{2}\right)}\, e^{-\frac{r^2}{4\nu t}}\left(\dfrac{r^2}{4\nu t}\right)^{\frac{\mathrm{Re}+2}{2}}, \end{cases} \tag{2.2}$$

where $\Gamma(a)$ is an Euler gamma function; $(a, z) = \int\limits_z^\infty x^{a-1}e^{-x}dx$ is an incomplete gamma function. We assume that the solution (2.2) extends also to $\mathrm{Re} < -2$. As is shown in [6], and as can be seen from the second formula of (2.2), it is a solution of the problem of rectangular vortex diffusion in the presence of a source only in the interval $-2 < \mathrm{Re} < \infty$. The vortex intensity in the interval $-\infty < \mathrm{Re} < -2$ rises, in accordance with (2.2). The reason for the erroneous nature of [4, 5] is that there the solution was obtained by passing to the limit $r_0 \to 0$ in the solution for $r_0 = \mathrm{const}$, where r_0 is the radius of a cylinder, enclosing a singularity at zero.

In accordance with [8], using the method of intermediate asymptotics, the solutions can be continued beyond the critical value of the parameter. We denote the linear dimension by a. We shall start from the equation for the vortex Ω_z:

$$\frac{1}{\nu}\frac{\partial \Omega_z}{\partial t} + (\mathrm{Re}-1)\frac{1}{r}\frac{\partial \Omega_z}{\partial r} = \frac{\partial^2 \Omega_z}{\partial r^2}. \tag{2.3}$$

We shall seek the solution of Eq. (2.3) in the form

$$\Omega_z = r^\alpha F(u), \tag{2.4}$$

where $u = r^2/(4\nu t)$. Substituting (2.4) into (2.3), we obtain

$$(2u)^2F'' + [(2\alpha - \mathrm{Re} + 2) + (2u)](2u)F' + \alpha(\alpha - \mathrm{Re})F = 0. \tag{2.5}$$

The general solution of Eq. (2.5) has the form

$$F = u^{-(2\alpha-\mathrm{Re}+2)/4}\, e^{-u/2} y(-(2\alpha - \mathrm{Re} + 2)/4,\ \mathrm{Re}/4,\ u), \tag{2.6}$$

where $y(k, m, x)$ is the solution of the Whittaker equation $4x^2y'' = (x^2 - 4kx + 4m^2 - 1)y$, i.e., the reduced form of a degenerate hypergeometric equation. To isolate the required solution from formula (2.6) we substitute the initial conditions $\gamma = \gamma_\infty$ or $\Omega = 0$ with $r \neq 0$. The final formulas have the form

$$\begin{cases} v_\theta = \dfrac{\gamma_\infty}{2\pi r}\left[1 - \dfrac{\Gamma\left(-\dfrac{\mathrm{Re}+2}{2},\ \dfrac{r^2}{4\nu t}\right)}{\Gamma\left(-\dfrac{\mathrm{Re}+2}{2}\right)}\left(\dfrac{r}{a}\right)^{\mathrm{Re}+2}\right], \\ \Omega_z = \dfrac{\gamma_\infty}{\pi a^2 \Gamma\left(-\dfrac{\mathrm{Re}+2}{2}\right)}\Gamma\left(-\dfrac{\mathrm{Re}}{2},\ \dfrac{r^2}{4\nu t}\right)\left(\dfrac{r}{a}\right)^{\mathrm{Re}}. \end{cases} \tag{2.7}$$

From the solution (2.7) it can be seen that the intensity of the vortex decreases not to zero, as in the case $\mathrm{Re} > -2$ [see (2.2)], but down to a steady-state vortical solution [6]

$$\begin{cases} \gamma(r) = \gamma_\infty\left[1 - \left(\dfrac{r}{a}\right)^{\mathrm{Re}+2}\right], \\ \Omega_z = -\dfrac{\mathrm{Re}+2}{2\pi a^2}\gamma_\infty\left(\dfrac{r}{a}\right)^{\mathrm{Re}}. \end{cases}$$

If we solve Eq. (2.1) with the initial condition $v_\theta = \gamma_\infty/(2\pi r)$ with Re < −2, not with an internal limiting cylinder but with a singularity at zero, by the method of a Laplace transform, then we obtain the solution (2.7). It can be shown by the same method that the asymptotic of any non-steady-state problem is a steady-state vortical solution with Re < −2. From the latter fact a conclusion could be drawn with respect to the mechanism of the vorticity intensification, on the basis of the instability of the flow of a viscous liquid, were it not for the following serious objection: At zero there is a singularity, which, as has been pointed out in several communications, can give rise to instability. Therefore, we return to the problem where there is no singularity.

§3. The model used for a description of vortical motions in the atmosphere (a tornado, waterspouts) is obtained from a general statement of the problem of Sec. 1, if for the principal flow we take the potential $\Phi = a(r^2/2 - z^2)$, describing the flow at the critical point, and substitute into Eq. (1.3)

$$\frac{\partial\gamma}{\partial t} + ar\frac{\partial\gamma}{\partial r} = \nu r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\gamma}{\partial r}\right). \tag{3.1}$$

We have the following boundary conditions:

at infinity

$$v_\theta = \gamma/2\pi r|_{r\to\infty} = 0; \tag{3.2}$$

at the axis

$$v_{\theta|r=0} = 0. \tag{3.3}$$

The steady-state solution of Eq. (3.1) with the boundary conditions (3.2), (3.3) depends on a, if $a \geq 0$, i.e., if the flow spreads out over a plane, then

$$\gamma = 0; \tag{3.4}$$

if $a < 0$, i.e., if there is ascending flow, then

$$\gamma = \gamma_\infty\left(1 - e^{ar^2/2\nu}\right). \tag{3.5}$$

The development of steady-state vortical solutions with determined values of the parameters in such situations is a characteristic factor [6]. The solution (3.5) was first found in [9] and is used by many authors, for example, for the construction of approximate models of a tornado or as an example of the existence of steady-state vortical motions in a viscous liquid.

Non-steady-state vortical solutions of Eq. (3.1) have not been found, probably for the following reasons. There is no self-similar solution for Eq. (3.1). A Laplace transform leads to an equation which can be solved numerically. Methods of intermediate asymptotics do not give the desired result. We apply a group analysis to Eq. (3.1) [10]. Using the infinitesimal operator admitted by (3.1) we obtain the characteristic solution for this equation; this operator has the form

$$X_\alpha = \frac{1}{a}e^{2at}\frac{\partial}{\partial t} + re^{2at}\frac{\partial}{\partial r}. \tag{3.6}$$

The operator (3.6) corresponds to the single-parametric continuous group of transforms

$$\begin{cases} t' = -\dfrac{1}{2a}\ln\left(e^{-2at} - 2\alpha\right), \\ r' = r\left(\dfrac{1}{1 - 2\alpha e^{2at}}\right). \end{cases} \tag{3.7}$$

Using the transform (3.7), from the solution (3.5) we construct the non-steady-state solution

$$\begin{cases} \gamma(r, t) = \gamma_\infty\left[1 - e^{\frac{a}{2\nu}r^2(1-2\alpha e^{2at})-1}\right], \\ \Omega_z(r, t) = -\dfrac{a\gamma_\infty}{\nu}\left(1 - 2\alpha e^{2at}\right)^{-1}e^{\frac{a}{2\nu}r^2(1-2\alpha e^{2at})-1} \end{cases} \tag{3.8}$$

The solution (3.8) can be used for the solution of different problems. If the main flow is descending $a > 0$, then it follows that $\exp(2at) \geq 1$ and $1 - 2\alpha \leq 0$. With $\alpha > 1/2$, i.e., with different initial given values of $\gamma(r, 0)$, expressions (3.7) and (3.8) tend toward zero when the time t tends toward infinity; i.e., they approach the steady-state solution (3.4). The special case where $\alpha = 0.5$ is a solution of the problem of rectilinear vortical filament diffusion in the presence of a descending flow above the plane $z = 0$. With an ascending main flow $a < 0$, $\exp(2at) \leq 1$ and $1 - 2\alpha \geq 0$; expression (3.8) tends toward the steady-state solution (3.5), differing from zero.

The special case obtained with $\alpha=0.5$ represents the diffusion of a vortical filament where the main flow is ascending. The vortical filament diffusion does not take place down to a zero value of the vorticity, but down to the steady-state solution (3.5). By choice of the parameter α, the initial data for $\gamma(r, 0)$ can be taken in such a way that

$$0 < |v_\theta(r, 0)| < \varepsilon \text{ and } 0 < |\Omega_z(r, 0)| < \varepsilon,$$

where ε is a previously given small number. For this, the value of α must satisfy the condition

$$|(\gamma_\infty - a)/[\nu(1 - 2\alpha)]| < \varepsilon^2.$$

In the latter case, the vorticity starts to be concentrated around the axis of rotation, down to the steady-state solution (3.5).

Since Eqs. (1.3), (3.1) are linear, the latter fact can be regarded from the point of view of hydrodynamic instability. This result argues that the flow of a viscous liquid with $|\Omega|=0$ and determined values of the parameters can be unstable with respect to vortical perturbations. An explanation of the localization and concentration of the vorticity in a viscous liquid can be based on the mechanism of the interaction between viscous and inertial forces, with which, with some values of the parameters of the flow, equilibrium is established between them in a vortical flow.

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